

**Spring 2015 Qualifier Exam:  
OPTIMIZATION**

**February 2, 2015**

**GENERAL INSTRUCTIONS:**

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

**SPECIFIC INSTRUCTIONS:**

Answer all 4 questions.

**POLICY ON MISPRINTS AND AMBIGUITIES:**

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. (a) Consider the linear programming problem of minimizing  $c^T x$  over a bounded polyhedron  $P \subseteq \mathbf{R}^n$ , subject to additional equality constraints  $a_i^T x = b_i, i = 1, 2, \dots, L$ . Assume that the feasible set for this problem is nonempty. Show that there exists an optimal solution that is a convex combination of at most  $L + 1$  extreme points of  $P$ .
- (b) Consider a modified version of the problem in (a) which is the same except for the addition of  $M$  further *inequality* constraints  $c_j^T x \leq d_j, j = 1, 2, \dots, M$ . Show that the solution is again expressible as a convex combination of the extreme points of  $P$ , involving at most  $\min(L + M + 1, K)$  extreme points, where  $K$  is the number of extreme points in  $P$ .
2. A farmer went to market and purchased a fox, a goose, and a bag of beans. On his way home, the farmer came to the right bank of a river and rented a boat. But in crossing the river by boat, the farmer could carry only himself and a single one of his purchases - the fox, the goose, or the bag of the beans. If left together, the fox would eat the goose, or the goose would eat the beans. The farmer's challenge was to carry himself and his purchases to the left bank of the river, leaving each purchase intact.

We formulate this as a binary programming problem the objective of which is to minimize the time required to complete the crossing. Here are the set and variable declarations for the model:

```

set      i      Items                /fox, goose, beans/,
        t      Crossings            /t0*t5/,
        a(i,i) Items to be kept apart /fox.geese, goose.beans/;

alias (i,j);

variable      TIME      Objective function (crossings to completion);

binary
variables     L(i,t)    Item i is on the left bank at the start of round t,
                R(i,t)    Item i is on the right bank at the start of round t,
                X(i,t)    Item i is in boat from right to left bank during round t,
                Y(i,t)    Item i is in boat from left to right bank during round t,
                B(i,t)    Item i is left on right bank during round t,
                Z(t)      Transportation is still in progress during round t;
```

The GAMS code produces a solution report as follows:

```

----      87 PARAMETER report  Summary report

                R                L                B                X                Y
```

t0.fox	1	1		
t0.goose	1		1	
t0.beans	1	1		
t1.fox	1		1	
t1.goose		1		1
t1.beans	1	1		
t2.fox		1		
t2.goose	1		1	
t2.beans	1			1
t3.fox		1		
t3.goose	1		1	
t3.beans		1		
t4.fox		1		
t4.goose		1		
t4.beans		1		
t5.fox		1		
t5.goose		1		
t5.beans		1		

Write down the formulation for this problem in terms of the data objects and variables above. You can express the objective and constraints in algebraic form, or in GAMS code, whichever you prefer. Your model should include constraints that determine the following conditions:

- (a) Which items are on the left bank at the start of cycle  $t + 1$ ?
  - (b) Which items are on the right bank at the start of cycle  $t + 1$ ?
  - (c) Which items are left behind on the right bank during cycle  $t$ ?
  - (d) Which items cannot be left on the right bank during cycle  $t$ ?
  - (e) Until the farmer is finished, selected items cannot be left together on the left bank.
3. Let  $n$  be a positive integer, and let  $b$  be a rational number with  $0 < b < n$ . Consider the mixed-integer set:

$$X = \left\{ x \in \mathbb{R}_+^n, y \in \{0, 1\}^n : \sum_{j=1}^n x_j \leq b, x_j \leq y_j, j = 1, \dots, n \right\}.$$

Let  $f = b - [b]$ . Define the inequalities:

$$\sum_{j \in S} x_j + f \sum_{j \in S} (1 - y_j) \leq b, \quad \forall S \subseteq \{1, \dots, n\} \text{ with } |S| = [b]. \tag{1}$$

- (a) Show that the inequalities (1) are valid for  $X$ . (Hint: consider two cases, either  $y_j = 1$  for all  $j \in S$ , or  $y_k = 0$  for some  $k \in S$ .)

- (b) Consider the special case with  $n = 2$  and  $b = 0.5$ . Write down one of the inequalities (1) for this case, and show that it is facet-defining.
  - (c) Considering again the case of general  $n$ , given a vector  $(\bar{x}, \bar{y}) \in \mathbb{R}_+^{2n}$ , formulate a binary integer program that can be used to find a violated inequality of the form (1) if one exists, or else shows no violated inequality exists.
  - (d) Explain why the binary integer program in part (c) can be solved as a linear program.
  - (e) Use the results of parts (c) and (d) to derive a compact linear programming formulation, in an extended variable space, which models the set of  $(x, y) \in \mathbb{R}_+^{2n}$  that satisfy the inequalities (1). The number of constraints and variables in this formulation should be linear in  $n$ . (Specifically, if  $P = \{(x, y) \in \mathbb{R}_+^{2n} : (1)\}$ , you should define a polyhedron  $Q$  in a higher variable space, having  $O(n)$  variables and inequalities, such that  $\text{proj}_{(x,y)}(Q) = P$ .)
4. Consider the continuously differentiable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , whose gradient has Lipschitz constant  $L$  (that is,  $\|\nabla f(y) - \nabla f(z)\| \leq L\|y - z\|$  for all  $y, z \in \mathbb{R}^n$ ). Suppose that  $f$  is bounded below on  $\mathbb{R}^n$  and in fact that  $f$  has the coercive property that  $f(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ . Consider an algorithm for minimizing  $f$  that generates an iteration sequence  $\{x^k\}_{k=0,1,2,\dots}$  according to the following formula:

$$x^{k+1} = x^k + \alpha_k p^k, \quad k = 0, 1, 2, \dots,$$

where  $\alpha_k \geq 0$  is selected to be a global minimizer of  $f$  along the direction  $p^k$  from  $x^k$ . The direction  $p^k$  has the form

$$p^k = - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial f(x^k)}{\partial x_{i_k}} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the nonzero element occurs in the  $i_k$  position, and the index  $i_k$  is chosen in such a way that

$$\left| \frac{\partial f(x^k)}{\partial x_{i_k}} \right| \geq \frac{1}{10} \|\nabla f(x^k)\|_\infty = \frac{1}{10} \max_{j=1,2,\dots,n} \left| \frac{\partial f(x^k)}{\partial x_j} \right|.$$

- (a) Show that for all  $\alpha > 0$ , we have

$$f(x^k + \alpha p^k) \leq f(x^k) + \alpha (p^k)^T \nabla f(x^k) + L\alpha^2 \|p^k\|_2^2.$$

- (b) Show that  $\lim_{k \rightarrow \infty} \nabla f(x^k) = 0$ .
- (c) Show that all accumulation points of this algorithm are stationary points of  $f$ .