Fall 2018 Qualifying Exam
OPTIMIZATION

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer all 4 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. **Linear Programming**

Recall the definitions of 1-norm and ∞-norm of a vector $x \in \mathbb{R}^n$:

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|, \qquad \|x\|_\infty := \max\{|x_i| : i = 1, \ldots, n\}.$$

Consider the following optimization problem:

$$\begin{aligned} &\min\ c^\top x \\ &\text{s.t.}\ \ \|Ax + b\|_1 \le 1. \end{aligned} \tag{1}$$

In this formulation, the decision variables are $x \in \mathbb{R}^n$, and the given data consists of $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

(a) Formulate this problem as a linear program (LP) in inequality form and prove that your LP formulation is equivalent to problem (1).
Hint 1: You may use additional variables.
Hint 2: Recall that you can show that two maximization problems (A) and (B) are equivalent by showing: (i) For any feasible solution of (A) there is a feasible solution of (B) with objective value not lower, (ii) For any feasible solution of (B) there is a feasible solution of (A) with objective value not lower.

(b) Derive the dual LP, and show that it is equivalent to the problem

$$\begin{aligned} &\max\ b^\top z - \|z\|_\infty \\ &\text{s.t.}\ \ A^\top z + c = 0. \end{aligned} \tag{2}$$

What is the relation between the optimal value of z of problem (2) and the optimal value of the variables in the dual LP derived?

(c) Let x be feasible for (1) (i.e., $\|Ax + b\|_1 \le 1$) and let z be feasible for (2) (i.e., $A^\top z + c = 0$). Using only the weak duality theorem, what can you argue about the relation between $c^\top x$ and $b^\top z - \|z\|_\infty$?

2. **Optimization Modeling**

An *economy* consists of *sectors*. You can think of a sector as a process that consumes resources at the start of the year and produces other resources at the end of the year. We can also choose an *activity level* for each sector, which determines how much consumption and production happens in each sector.

Example. Here is an example with two sectors and three resources:

- Sectors: $\{\text{house-building}, \text{road-building}\}$
- Resources: $\{\text{wheat}, \text{brick}, \text{ore}\}$

At an activity level of 1, suppose we have the following:

- House-building consumes (1 brick, 1 ore) and produces (2 wheat, 2 brick, 2 ore).
- Road-building consumes (1 wheat, 1 brick) and produces (1 ore, 2 brick).

We can think of the consumption and production levels above as *rates*. To find the actual consumption and production, we multiply by the activity level. For example, if we choose an activity level of 100 for house-building and 50 for road-building in Year 1, then our economy behaves as follows:

- Total resources consumed by all sectors: (50 wheat, 150 brick, 100 ore)
- Total resources produced by all sectors: (200 wheat, 300 brick, 250 ore)

Every year, we must choose activity levels for the sectors, with the goal of making every sector *grow*. That is, we want the activity level of each sector to increase compared to the previous year. However, we cannot grow too fast: the consumption of a given year cannot exceed the production in the previous year. For example, if we decided to triple our activity levels for Year 2, this would require consuming (150 wheat, 450 brick, 300 ore). This is not possible because we produced an insufficient amount of brick and ore (300 and 250 respectively) in the previous year.

For this problem, we will look at optimizing growth over a two-year planning period. Specifically, we assume:

- There are m resources $i = 1, \dots, m$ and n sectors $j = 1, \dots, n$.
- For sector j, resource i is consumed at a rate b_{ij} and produced at a rate a_{ij}. These are fixed quantities known ahead of time.
- Sector j has activity level $x_j^{(1)}$ in Year 1 and $x_j^{(2)}$ in Year 2. These activity levels are things we must decide on.

- Year 1 activity levels are strictly positive and Year 2 activity levels are nonnegative.
- The consumption in Year 2 must not exceed the production in Year 1.

Define the *growth rate* of sector j as $x_j^{(2)}/x_j^{(1)}$. Our objective will be to maximize the *minimum growth rate*, which is the growth rate of the sector with the smallest growth rate. This ensures that every sector is growing. Finally, here are the problems:

(a) Formulate the above as an optimization problem. That is, specify the parameters, decision variables, constraints, and objective function.

(b) The formulation from Part (a) includes the strict inequalities $x_j^{(1)} > 0$ for $j = 1, \ldots, n$. Explain why strict inequalities are generally undesirable in an optimization model. Also explain how and why the strict inequalities can be replaced by $x_j^{(1)} \geq 1$ without any loss of generality.

(c) Suppose we want to know whether it's possible to achieve a minimum growth rate of r. Explain how the problem of Part (a) can be reformulated as a linear program where r appears as a parameter.

(d) Consider maximizing the *total annual growth* $\left(\sum_{j=1}^n x_j^{(2)}\right)/\left(\sum_{j=1}^n x_j^{(1)}\right)$ instead. In this scenario, we allow activity levels in Year 1 to be zero, so long as the total activity in Year 1 is strictly positive. Formulate this problem as a linear program.

3. **Integer Optimization**

Let $S = \{v^1, \dots, v^T\} \subseteq \mathbb{R}^n$ be a finite (possibly huge) set of points and consider the optimization problem:

$$\begin{aligned} z^* = \min\ & c^\top x && (3) \\ \text{s.t. } & Ax \geq b && (4) \\ & x \in S && (5) \end{aligned}$$

where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. Assume that this optimization problem has a feasible solution, so that z^* is finite. For $\lambda \in \mathbb{R}^m_+$, define:

$$\begin{aligned} z(\lambda) = \lambda^\top b + \min\ & (c^\top - \lambda^\top A)x \\ \text{s.t. } & x \in S \end{aligned}$$

Finally, define:

$$z^{LD} = \max\{z(\lambda) : \lambda \in \mathbb{R}^m_+\}$$

The number of points that will be allocated to each part when grading are given in brackets at the beginning of each part.

(a) [2 pts] Show that $z(\lambda) \leq z^*$ for any $\lambda \in \mathbb{R}^m_+$.

(b) [4 pts] Recall that $\text{conv}(S)$ is notation for the convex hull of S. Show that

$$\begin{aligned} z^{LD} = \min\ & c^\top x \\ \text{s.t. } & Ax \geq b \\ & x \in \text{conv}(S). \end{aligned}$$

[Hint: Start by formulating the problem $\max\{z(\lambda) : \lambda \in \mathbb{R}^m_+\}$ as a linear program, possibly by adding additional decision variable(s).]

(c) [2 pts] Provide an example where $z^{LD} < z^*$. [Hint: this can be done with $n = 1$ and a set S containing two points.]

(d) [1 pt] Suppose $S = \{x \in \mathbb{Z}^n_+ : Dx \geq d\}$. Use the result from part (b) to show that the $z^{LD} \geq z^{LP}$, where z^{LP} is the optimal value of the linear programming relaxation of (3)-(5):

$$\begin{aligned} z^{LP} = \min\ & c^\top x \\ \text{s.t. } & Ax \geq b \\ & Dx \geq d \\ & x \in \mathbb{R}^n_+ \end{aligned}$$

(e) [1 pt] Using the definition of S from part (d), suppose that the matrix D is totally unimodular and d is an integer vector. Again using results from previous parts, argue that in this case $z^{LD} = z^{LP}$.

4. **Nonlinear Optimization**

Consider the inexact Newton method applied to the problem of minimizing a twice Lipschitz continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. The search direction p_k at iteration $k = 0, 1, 2, \ldots$ satisfies the formula

$$H_k p_k + g_k = r_k,$$

where $H_k = \nabla^2 f(x_k)$, $g_k = \nabla f(x_k)$, and the residual vector r_k satisfies

$$\|r_k\| \le \eta_k \|\nabla f(x_k)\|,$$

for some $\eta_k \in [0, \bar{\eta}]$, where $\bar{\eta} \in [0, 1)$.

Assume that there is a local minimizer x^* at which second-order sufficient conditions are satisfied, that is, $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite. Assume that the smallest eigenvalue of $\nabla^2 f(x^*)$ is μ and the largest is L, with $0 < \mu < L$.

Consider the full-step version of the method, in which we define $x_{k+1} = x_k + p_k$ (so that $g_{k+1} = \nabla f(x_{k+1})$).

(a) Show that there are positive quantities $\rho > 0$ and $M_1 > 0$ and $M_2 > 0$ such that if $\|x_k - x^*\| \le \rho$, we have that

$$\|p_k\| \le M_1 \|g_k\|, \quad \frac{\|g_{k+1}\|}{\|g_k\|} \le \eta_k + M_2 \|g_k\|.$$

(You do not need to find specific values for ρ, M_1, and M_2, just explain why such quantities must exist.)

(b) Consider the Taylor series expansion

$$f(x_{k+1}) = f(x_k) + g_k^T p_k + \frac{1}{2} p_k^T H_k p_k + O(\|p_k\|^3).$$

Express the sum of the first- and second-order terms $\left(g_k^T p_k + \frac{1}{2} p_k^T H_k p_k\right)$ in terms of H_k^{-1}, g_k, and r_k.

(c) Using the result of (b), find a condition on r_k of the form $\|r_k\| \le T\|g_k\|$ (for some $T > 0$) that guarantees $g_k^T p_k + \frac{1}{2} p_k^T H_k p_k \le 0$ when x_k is sufficiently close to x^*. Express T in terms of L, μ.

(d) Similarly to part (c), find a value $\bar{T}$ such that when $\|r_k\| \le \bar{T}\|g_k\|$ and x_k is sufficiently close to x^*, p_k is a descent direction for f, that is, $g_k^T p_k < 0$.