Spring 2016 Qualifier Exam:
OPTIMIZATION

February 1, 2016

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer all 4 questions to the best of your abilities.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Let k be an integer with $1 \leq k < n$. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be defined by

$$f(x) = \sum_{i=1}^{k} x_{[i]},$$

where $x_{[i]}$ is the ith largest value in the vector x.

   (a) Show that f is convex.

   (b) Show that for any $x \in \mathbb{R}^n$, we have

$$f(x) = \max_{y} x^T y \text{ s.t. } \sum_{j=1}^{n} y_j = k,\ 0 \leq y_i \leq 1, i = 1, 2, \ldots, n.$$

   (c) For any $\alpha \in \mathbb{R}$ show that $f(x) \leq \alpha$ if and only if there exist $\lambda \in \mathbb{R}^n_+$ and $u \in \mathbb{R}$ such that

$$ku + \sum_{j=1}^{n} \lambda_j \leq \alpha,\ u + \lambda_i \geq x_i, i = 1, 2, \ldots, n.$$

   (d) Prove that a solution to

$$\max_{x} \sum_{i=1}^{n} x_i \text{ s.t. } f(x) \leq \alpha$$

   is $x_i = \frac{\alpha}{k}$, $i = 1, 2, \ldots, n$.

2. Given an undirected, connected, not necessarily complete graph $G = (V, E)$, with $|V| = n$, $|E| = m$, and a weight vector $w \in \mathbb{R}^m$, we are interested in formulating optimization models for finding *spanning trees* of G with specific properties. Answer the following questions:

   (a) How many edges are in each spanning tree of G?

   (b) Using (only) the decision variables $x_e \in \mathbb{Z}\ \forall e \in E$, write an integer programming model that finds a *minimum weight* spanning tree of G.

   (c) Write an integer programming formulation that maximizes the number of nodes in the spanning tree that have degree exactly 2. You may use additional decision variables. And the following jibberish may come in handy for you:

   - $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b \Leftrightarrow \sum_{j \in N} a_j x_j + M\delta \leq M + b$
   - $\sum_{j \in N} a_j x_j \leq b \Rightarrow \delta = 1 \Leftrightarrow \sum_{j \in N} a_j x_j - (m - \epsilon)\delta \geq b + \epsilon$
   - $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \geq b \Leftrightarrow \sum_{j \in N} a_j x_j + m\delta \geq m + b$
   - $\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1 \Leftrightarrow \sum_{j \in N} a_j x_j - (M + \epsilon)\delta \leq b - \epsilon$

   (d) Of course, there may be *many* spanning trees that have the maximum number of degree 2 nodes. Discuss how you might find, out of all of these, the tree with the maximum number of degree 2 nodes that has minimum weight with respect to the weight vector w.

   (e) A graph is connected if and only if there is a flow of value one from each node $v \in V \setminus \{r\}$ to an arbitrary *root* node $r \in V$. Use this fact to write another formulation for the minimum-weight spanning tree problem. You may use decision variables besides x_e.

3. Let

$$S := \{x \in \{0,1\}^4 : 90x_1 + 35x_2 + 26x_3 + 25x_4 \le 138\}.$$

(a) Show that

$$S = \{x \in \{0,1\}^4 : 2x_1 + x_2 + x_3 + x_4 \le 3\},$$

and

$$\begin{aligned} S = \{x \in \{0,1\}^4 : 2x_1 + x_2 + x_3 + x_4 &\le 3 \\ x_1 + x_2 + x_3 &\le 2 \\ x_1 + x_2 + x_4 &\le 2 \\ x_1 + x_3 + x_4 &\le 2\}. \end{aligned}$$

(b) Can you rank these three formulations in terms of the tightness of their linear relaxations, when $x \in \{0,1\}^4$ is replaced by $x \in [0,1]^4$? Show any strict inclusion.

4. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function with uniformly Lipschitz continuous gradient. That is, there exist constants γ and L with $0 < \gamma < L$ such that

$$f(y) \le f(x) + \nabla f(x)^T (y - x) + \frac{L}{2}\|y - x\|_2^2, \tag{1a}$$

$$f(y) \ge f(x) + \nabla f(x)^T (y - x) + \frac{\gamma}{2}\|y - x\|_2^2, \tag{1b}$$

for all $x, y \in \mathbb{R}^n$.

(a) Prove that f attains its minimizer at a unique point x^*, with $\nabla f(x^*) = 0$. (You may use the fact that if f is bounded below over a compact set C, it attains its minimum on C, that is, there is a point $x^* \in C$ such that $f(x^*) = \inf_{y \in C} f(y)$.)

(b) Consider the short-step steepest descent procedure for minimizing f:

$$x^{k+1} = x^k - \alpha \nabla f(x^k), \quad \text{where } \alpha \equiv 1/L.$$

Prove that the sequence $\{f(x^k)\}_{k=0,1,\ldots}$ converges to $f(x^*)$ at a linear rate, in particular,

$$[f(x^{k+1}) - f(x^*)] \le \left(1 - \frac{\gamma}{L}\right)[f(x^k) - f(x^*)], \quad k = 0, 1, 2, \ldots. \tag{2}$$

Hint: Use (1a) to prove that

$$f(x^{k+1}) \le f(x^k) - \frac{1}{2L}\|\nabla f(x^k)\|^2,$$

and use (1b) with $x = x^k$ to show that

$$f(x^*) \ge f(x^k) - \frac{1}{2\gamma}\|\nabla f(x^k)\|^2.$$

(c) Prove that the same convergence rate holds if we obtain x^{k+1} by an exact line search along $-\nabla f(x^k)$, that is,

$$x^{k+1} = x^k - \alpha_k \nabla f(x^k), \quad \text{where } \alpha_k = \arg\min_{\alpha} f(x^k - \alpha \nabla f(x^k)).$$