Spring 2014 Qualifier Exam:
OPTIMIZATION

February 3, 2014

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 4 out of 5 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. You have a set J of jobs that must be scheduled within a set of time periods $\mathcal{T} = \{1, \ldots, T\}$. Each job has an integer processing time $p_j > 0$. When being processed, jobs use capacity from a set I of machines. In particular, when job $j \in J$ is being processed it requires $a_{ij} > 0$ units of the capacity of machine $i \in I$. At any point in time, the total available capacity of machine $i \in I$ is denoted by $b_i > 0$. Jobs can be processed simultaneously, provided the machine capacity constraints are not exceeded, but jobs cannot be interrupted (i.e., once a job starts it will be in process for p_j consecutive time periods). In the following questions, you should use (at least) the following "time-indexed" decision variables to determine the start-time of the jobs.

   - $x_{jt} = 1$ if job $j \in J$ starts at time $t \in \{1, 2, \ldots, T - p_j\}$, $x_{jt} = 0$ otherwise.

   (a) Write a linear integer programming formulation to minimize the sum of start times of the jobs.

   (b) Suppose now each job $j \in J$ has an earliest start-time $r_j \geq 1$ and a latest completion time $D_j \leq T$. One possible way to model these restrictions is with the constraints:

   $$r_j \leq \sum_{t \in \mathcal{T}} t x_{jt} \leq D_j - p_j, \quad j \in J. \tag{1}$$

   However, these constraints would lead to a weak linear programming relaxation. Provide a *different* way to model the start-time and completion time restrictions, and argue why your model is preferred.

   (c) Now suppose that you find it is not feasible to schedule all the jobs so that they are completed by their latest completion time D_j. Thus, you wish to relax this constraint, and instead penalize lateness in the objective. Specifically, if job $j \in J$ completes at time $t > D_j$, then this will be penalized by $(D_j - t)^2$. Modify your model to replace the objective with the objective of minimizing the sum of penalties. (Your model must remain an integer *linear* program.)

   (d) Now consider a pair of jobs j and k that have a precedence relationship: job k cannot begin processing until job j completes its processing. This can be modeld with the constraint

   $$\sum_{t \in \mathcal{T}} t x_{jt} + p_j \leq \sum_{t \in \mathcal{T}} t x_{kt}. \tag{2}$$

   However, this constraint again leads to a weak linear programming relaxation. Provide an alternative model for this restriction that would lead to a better relaxation. (Hint: the formulation should involve inequalities that only have coefficients 0 or 1 on the decision variables). You *do not* have to provide a proof that the LP relaxation of your formulation is better than (2).

2. Consider the parametric linear program PLP(θ):

$$\begin{array}{rlllllll} z(\theta) = \min & 4x_1 & +2x_2 & & +x_4 & & & \\ \text{s.t.} & x_1 & & -x_3 & +x_4 & & & =\theta \\ & x_1 & +x_2 & & & & & =\theta \\ & x_1 & & -x_3 & & -x_5 & & =1 \\ & & x_2 & & & & +x_6 & =1 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \geq 0 \end{array}$$

(a) Write the dual linear program of PLP(θ).

(b) Determine $z(\theta)$ $\forall \theta \in \mathbb{R}$.

(c) Is $z(\theta)$ a convex function, a concave function, or neither?

(d) Is your answer from part (c) generally true? That is, for the general parametric linear program, where the right-hand side vector b is a treated as a parameter, is the value function $z(b)$,

$$z(b) \stackrel{\text{def}}{=} \min_{x \geq 0} \{c^\top x \mid Ax = b\},$$

a convex function of b, a concave function of b, or neither? Provide a proof of your claim.

(e) Let $x_4^*(\theta)$ be an optimal value for the decision variable x_4^* as a function of the parameter θ in PLP(θ). (For PLP(θ), $x_4^*(\theta)$ is a single-valued mapping (a function)). Determine $x_4^*(\theta)$ $\forall \theta \in \mathbb{R}$.

(f) Is $x_4^*(\theta)$ a convex function, a concave function, or neither?

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3. Let $S = \{x^1, x^2, \ldots, x^t\}$ be a finite set of points in $\mathbf{R}^n$, and let $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ be a non-empty polytope where A is an $m \times n$ matrix and $b \in \mathbf{R}^n$. Let $\hat{x} \in \mathbf{R}^n$ be given. Formulate a linear program that determines whether or not $\hat{x} \in \text{conv}(S \cup P)$, and if not identifies a valid inequality for $\text{conv}(S \cup P)$ that is violated by $\hat{x}$. State how the linear program answers the question and prove that it provides a correct answer (this requires proving something in its two possible statements: $\hat{x} \in \text{conv}(S \cup P)$ and $\hat{x} \notin \text{conv}(S \cup P)$).

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4. Assume that X is a bounded polyhedron. Let $x^0 \in X$ be given and for $k \geq 0$ let $\bar{x}^k$ be defined as an extreme point of X satisfying

$$\bar{x}^k \in \arg\min_{x \in X} \nabla f(x^k)^T (x - x^k),$$

and suppose that x^{k+1} is a stationary point of the optimization problem

$$\min_{x \in X^k} f(x),$$

where X^k is the convex hull of x^0 and the extreme points $\bar{x}^0, \ldots, \bar{x}^k$.

   (a) Write down the definition of X^k explicitly.

   (b) Define the notion of a stationary point for $\min_{x \in X} f(x)$.

   (c) Under what conditions is $\bar{x}^k$ well defined. Identify a (known) algorithm that can determine $\bar{x}^k$.

   (d) Show that there exists a finite integer k such that the above method finds a stationary point of f over X.

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5. (a) Consider the following semidefinite program, in standard form:

$$\min_{X \in S\mathbb{R}^{2\times 2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet X \text{ subject to } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X = 1, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bullet X = 0, \quad X \succeq 0.$$

   Write down the dual of this problem, and find the complete primal and dual solution sets, together with the optimal objective value for both problems.

   (b) Consider the following semidefinite program, in standard form:

$$\min_{X \in S\mathbb{R}^{2\times 2}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \bullet X \text{ subject to } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X = 0, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet X = 2, \quad X \succeq 0.$$

   Write down the dual of this problem, and find the complete primal and dual solution sets, together with the optimal objective value for both problems.

   (c) Give sufficient conditions on a primal-dual pair of semidefinite programs that guarantees that the solutions sets of both problems are *nonempty* and *bounded* and have equal objective value. Are these conditions satisfied by the problems in parts (a) and (b)?