Please answer all four questions below.

1. Show that the following problem is NP-complete: Given positive integers \( n, n_1, \ldots, n_k \) in binary representation, decide whether \( n \) can be written as a product \( \prod_{i=1}^{k} n_i^{e_i} \) where the \( e_i \)'s are nonnegative integers.

2. You are playing a video game with \( n \) levels. The goal is to pass all of the levels and reach the finish line without accumulating too many “demerit points”. In each level \( \ell \), you have \( k \) actions available. Action \( i \) in level \( \ell \) brings you \( d_{\ell,i} \) demerit points and has a probability \( p_{\ell,i} \) of succeeding. If the action succeeds, you pass this level and move on to level \( \ell + 1 \). However, if it fails, you are sent back to level 1 and must solve all of the levels again. Given the quantities \( d_{\ell,i} \) and \( p_{\ell,i} \) for each pair \((\ell, i)\), design an algorithm that runs in time \( \text{poly}(n, k) \) and finds a strategy for playing the game that minimizes the expected number of demerit points accumulated before the game ends.

3. The feedback vertex set problem is as follows: We are given an undirected graph \( G = (V, E) \), and our goal is to find the smallest subset of vertices whose removal makes the graph acyclic. The “dual” to this problem asks to find the largest set of vertex-disjoint cycles in the graph \( G \).
   
   (a) Prove that the size of any feasible solution to the dual problem provides a lower bound on the size of any feasible solution to the feedback vertex set problem.
   
   (b) Given a feasible solution to the dual problem, consider the set of vertices in the union of all cycles in the dual solution. Prove that if the dual solution is maximal, then this set of vertices is a feedback vertex set. What approximation to the feedback vertex set problem does this imply?
   
   (c) Consider the following algorithm for constructing a primal and dual solution in tandem: while the graph contains a cycle, pick one such cycle \( C \); Include the cycle \( C \) in the dual solution; Remove a subset \( S \) of the vertices in the cycle \( C \) from the graph and include this subset in the primal solution; Iterate.
   
   Figure out how to choose the cycle \( C \) and the subset \( S \) in each iteration so as to obtain a good approximation for the feedback vertex set problem. Aim for a bound of \( O(\log n) \) on the approximation ratio, where \( n \) is the number of vertices in \( G \).

   *Hint: You may want to use the fact that every \( n \)-vertex graph with minimum degree 3 contains a cycle of length at most \( 2\log n \).*

4. For \( k = 0, 1, 2, \ldots \), let \( \Delta_k^c := \text{DTIME}(2^{O(n)})^{2\log n} \). This is called the linear-exponential hierarchy. Show that for some \( k \in \{0, 1, 2, \ldots\} \), \( \Delta_k^c \) contains a language that has maximum circuit complexity at every input length. Pick \( k \) as small as possible.

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\(^1\text{That is, no more cycles can be added to the solution without violating vertex-disjointness.}\)